MATRICES (REVIEW FROM LINEAR ALGEBRA)

- An m×n ("m by n") matrix A over a set S is a rectangular array of elements of S arranged into m rows and n columns:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \leftarrow i^{th} \text{ row}$$

$\uparrow j^{th}$ column

- We write $A = (a_{ij})$
- The a_{ij} entry of the A matrix is called the ijth entry of A
- A matrix with the same number of rows and columns (m=n) is called a square matrix. The main diagonal of a square matrix of size n×n consists of all the entries $a_{11}, a_{22}, \ldots, a_{nn}$

- Let $A=(a_{ij})$ be an m×k matrix and $B=(b_{ij})$ a k×n matrix with real entries. The (matrix) product of A times B, denoted AB is the m×n matrix $C=(c_{ij})$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ik} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mk} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1j} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2j} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kj} & \cdots & b_{kn} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1j} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2j} & \cdots & c_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{i1} & c_{i2} & \cdots & c_{ij} & \cdots & c_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mj} & \cdots & c_{mn} \end{bmatrix}$$

  Where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{ik}b_{kj} = \sum_{r=1}^{k} a_{ir}b_{rj}$
  for all i from 1 to m and j from 1 to n

- Matrix multiplication is **associative** but **not commutative**.
- For any positive integer n, the identity matrix I_n is the n×n matrix where all the entries are 0 except for the main diagonal entries which are all 1.
- Identity matrices **work as identity elements** in matrix multiplication: if $A=(a_{ij})$ is an m×n matrix, then $I_m \times A = A \times I_n = A$
- For any n×n matrix A, the powers of A are defined as follows:

  $A^0 = I_n$ $\qquad A^n = AA^{n-1}$ $\qquad$ for all integers $n \geq 1$

ADJACENCY MATRIX OF A GRAPH

- Let G be a directed graph with ordered vertices $v_1, v_2, \ldots, v_n$.
  The adjacency matrix of G is the n×n matrix $A = (a_{ij})$ such that
  for i and j from 1 to n, a_{ij}=the number of arrows from v_i to v_j.
- Let G be an undirected graph with ordered vertices $v_1, v_2, \ldots, v_n$.
  The adjacency matrix of G is the n×n matrix $A = (a_{ij})$ such that
  for i and j from 1 to n, a_{ij}=the number of edges connecting v_i and v_j.
- Note that the adjacency matrix of an undirected graph is symmetric,
  i.e. for any i and j from 1 to n, $a_{ij} = a_{ji}$

GRAPHS PROPERTIES

Edgeless graph

- The adjacency matrix of an edgeless graph is a zero matrix.

Complete graph

- The adjacency matrix of a complete graph is such that all entries are 1 except for the main diagonal entries which are all 0

Connected components

- Let G be a graph with connected components $G_1, G_2, \ldots, G_k$. If each connected component G_i has n_i vertices **numbered consecutively**, then the adjacency matrix of G has the form

$$\begin{bmatrix} A_1 & O & \cdots & O & \cdots & O \\ O & A_2 & \cdots & O & \cdots & O \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ O & O & \cdots & A_i & \cdots & O \\ \vdots & \vdots & & \vdots & \ddots & \vdots \\ O & O & \cdots & O & \cdots & A_k \end{bmatrix}$$

  Where each A_i is the $n_i \times n_i$ adjacency matrix of G_i and the 0's represent matrices whose entries are all 0.

COUNTING WALKS OF LENGTH N

- Let G be a graph with ordered vertices $v_1, v_2, \ldots, v_m$
  and A be the adjacency matrix of G, then

  for each positive integer n and for all integers i,j from 1 to m,
  the ij^{th} entry of A^n = the number of walks of length n from v_i to v_j.